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The downstream flow beyond an obstacle

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This paper is concerned with theoretical predictions, given the upstream conditions from a rigid obstacle of arbitrary shape, of the downstream flow beyond the obstacle for an incompressible inviscid fluid sheet under the action of gravity. The fluid sheet flows upstream over a level bottom, continues to flow over (or under) an obstacle leading to a downstream region over a level bottom. In the absence of surface tension, a nonlinear steady-state solution of the problem is used to predict the downstream values of the free-surface wave height for the full range of the far-upstream Froude number. The general results obtained are then applied to a special case of fluid flowing over a stationary hump leading to a supercritical flow far downstream and detailed numerical comparison is made with available experimental results, with very good agreement.

1. Introduction

This paper is concerned with steady two-dimensional flow of an incompressible inviscid fluid sheet under the action of gravity past a rigid obstacle of arbitrary shape. The obstacle may be a hump over a bottom, a moving object at the top free surface, or even a partially submerged object such as a sluice gate. The fluid sheet flows upstream over a level bottom (labelled I in figure 1), continues to flow over (or under) a region occupied by the obstacle (II in figure 1) leading to a downstream region (III in figure 1). In both the upstream and downstream regions from the obstacle, the bottom surfaces are assumed to be level and the free top surfaces subjected to a constant atmospheric pressure. In the absence of the effect of surface tension and for given upstream conditions from the obstacle, the nonlinear differential equation for the wave height in the downstream region is integrated leaving only one undetermined constant S_3. It is then shown that S_3 possesses upper and lower bounds beyond which values a steady-state solution cannot exist in the downstream region. These bounds on S_3 play a significant role when the application of the theory is considered in §5.

In §2, we briefly consider a nonlinear differential equation (see (2.3)) for the free-surface wave height h appropriate for a two-dimensional steady motion confined to the (x, z)-plane of the rectangular Cartesian coordinate system (x, y, z). This differential equation can be derived from the system of basic equations of a direct theory of constrained fluid sheets (Green & Naghdi 1976*a*, 1977), and special cases of this differential equation can be brought into correspondence with those derived and used in a number of related previous studies, notably by Benjamin & Lighthill (1954), Benjamin (1956), Green & Naghdi (1976*a*) and Naghdi & Rubin (1981). Also a brief comparison is made in §2 between the differential equation (2.3) of the present paper and the corresponding Boussinesq equation and Saint-Venant equation for steady flows.

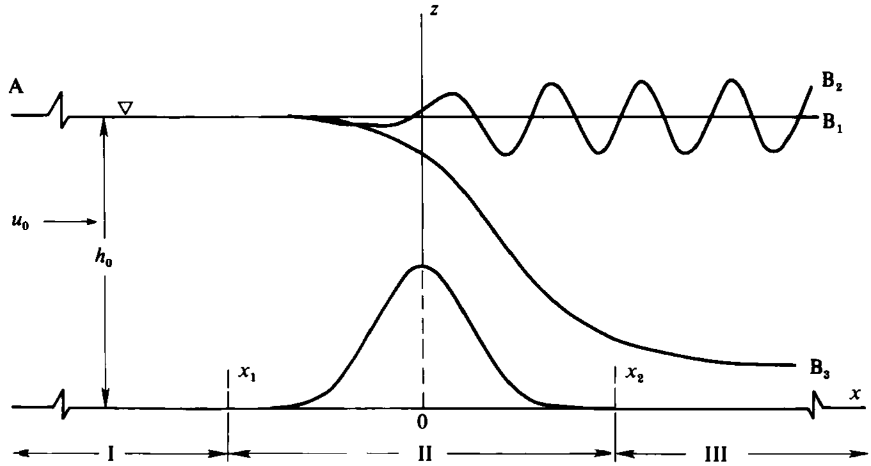

FIGURE 1. A sketch of fluid flow past an obstacle in the (x, z)-plane of a rectangular Cartesian coordinate system showing the upstream, the transition and the downstream regions, labelled I, II, III, respectively. Also shown are the vertical height h_0 and horizontal velocity u_0 far upstream. Depending on the type of obstacle occupying the transition region II (for example, the shape of the hump and the maximum height of the bottom surface of the fluid sheet), the downstream flow may approach: (a) a uniform flow of height h_0 and velocity u_0, i.e. AB_1, (b) a standing cnoidal wave, i.e. AB_2, or (c) a uniform flow of diminishing height, i.e. AB_3.

The problem, as stated in the first paragraph of this section, is formulated in §3 and this is followed with a brief discussion of the upstream flow. Assuming that the far upstream is at a uniform height h_0 and moving with constant velocity u_0, the flow in the upstream region of the obstacle can be completely characterized in terms of h_0 and the far-upstream Froude number F defined by

$$F^2 = u_0^2/gh_0, \tag{1.1}$$

where g denotes the constant gravitational acceleration. Depending on whether $F^2 < 1$, $= 1$, or > 1, the far-upstream flow is referred to as subcritical, critical, or supercritical, respectively.

Regardless of the shape of the rigid obstacle, and on the assumption that there is no mass loss or energy loss, with the use of the differential equation for the wave height h derived in §2, it is shown in §4 that the downstream flow is completely determined if the value of one of the constants of integration, namely S_3, is known. For given upstream conditions, the constant S_3 can be determined from h and its gradient at any one point in the downstream region. In particular, for given far-upstream values of information h_0 and $F^2 < 1$, a steady flow is possible in the downstream region only if S_3 is bounded from above and below by the values S_3^+ and S_3^- given by (4.7) and (4.10) respectively [see also (4.15) and (4.16)], where a parameter $\gamma = \frac{1}{4}F^2[1+(1+(8/F^2))^{\frac{1}{2}}]$ occurs only in the value of S_3^-. For the upper-bound value S_3^+, the downstream flow is subcritical with a uniform depth h_0 everywhere. On the other hand, for the lower-bound value S_3^-, the downstream flow is supercritical, approaching a uniform flow of depth γh_0 and a Froude number F^2/γ^3 far downstream. For a value of $S_3^- < S_3 < S_3^+$, the downstream flow has the form of stationary cnoidal waves. Somewhat similar conclusions can be arrived at when the far-upstream information is h_0 and $F^2 > 1$ or $F^2 = 1$, but these cases are not dealt with here.

In §5, we deal with a special case of our general developments of §4 by considering a fluid sheet flowing over a stationary bell-shaped hump. Making use of the fact that the constant S_3 for the downstream flow is bounded from above and below, it can be demonstrated that a steady-state flow is not possible if the upstream fluid height is below a certain value. Furthermore, when the parameters describing the shape of the obstacle are chosen so that the obstacle is identical with that in the experiment of Sivakumaran, Tingsanchali & Hosking (1983), the calculated results for minimum values of h_0 that render the downstream flow supercritical are found to be in very good agreement with the experimental data.

2. Differential equation for the free-surface wave height

For steady two-dimensional flow of an incompressible, homogeneous, inviscid fluid sheet under the action of gravity we obtain, in this section, a nonlinear differential equation for the free-surface wave height h over a variable bottom of arbitrary profile. We use the basic equations of the restricted theory of a directed fluid sheet (Green & Naghdi 1976*a*, 1977), to derive a differential equation which is utilized in the rest of the paper. A comparison will be indicated later in this section between the nature of this differential equation and other such differential equations used in the current literature for applications to problems similar to that discussed in §5.

In order to maintain closer continuity with some of the references cited (e.g. Benjamin & Lighthill 1954; Benjamin 1956), we adopt here a somewhat different notation from those utilized previously (e.g. Green & Naghdi 1976*a*, *b*, 1977). Let $x_i = (x, y, z)$, $(i = 1, 2, 3)$, be a set of fixed rectangular Cartesian-coordinate axes and denote by $\boldsymbol{e}_i = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the associated orthonormal base vectors. Further, for two-dimensional flow confined to the (x, z)-plane, let $u = u(x)$ denote the horizontal component of the velocity, $h = h(x)$ the free-surface wave height, $H = H(x)$ the vertical distance to the variable bottom surface of the fluid, ρ^* the constant mass density of incompressible fluid, and $\hat{p}$ the pressure at the top surface. Also, in the present paper, we assume that the far-upstream flow approaches that of a uniform flow, i.e.

$$u \to u_0 = \text{const.}, \quad h \to h_0 = \text{const.}, \quad H \to 0 \quad \text{as } x \to -\infty. \tag{2.1}$$

Then, for an incompressible fluid, the conservation of mass can be reduced to

$$hu = Q = h_0 u_0, \tag{2.2a, b}$$

where Q is a constant which has been determined from the boundary condition (2.1). [Our present notations $u, h, H, \rho^*, \hat{p}, Q$ correspond respectively to $u, \phi, \alpha, \rho^*, \hat{p}, k$ of a number of previous papers, e.g. Green & Naghdi 1977, Naghdi & Rubin 1981]. Next, with $\hat{p}_\infty$ denoting the top-surface pressure at $x = -\infty$ and after using (2.2*b*), the equations of motion of the theory can be reduced to a Bernoulli-type integral of the form

$$\frac{\hat{p}}{\rho^*} + \tfrac{1}{2}u^2 + g(h + H - h_0) + \chi(h, h_x, h_{xx}; H_x, H_{xx}) = \frac{\hat{p}_\infty}{\rho^*} + \tfrac{1}{2}u_0^2, \tag{2.3}$$

with

$$\chi = Q^2\left\{\frac{1}{3}\frac{h_{xx}}{h} - \frac{1}{6}\left(\frac{h_x}{h}\right)^2 + \frac{1}{2}\left[\left(\frac{H_x}{h}\right)^2 + \frac{H_{xx}}{h}\right]\right\}. \tag{2.4}$$

If the bottom surface of the fluid is level ($H = H_x = H_{xx} = 0$) and $\hat{p}$ is taken to be the constant atmospheric pressure, then the equation resulting from (2.3) after substitution for u from (2.2*a*) and after an integration can be brought into correspondence with one given by Green & Naghdi (1976*b*, §7), or with that derived earlier by Benjamin & Lighthill (1954) from the three-dimensional equations for long waves.

The Boussinesq equations and the corresponding equations in the theory of Green & Naghdi (1976*a*, *b*, 1977) for unsteady flow have been compared previously (Naghdi 1979, p. 514), and a further discussion on the subject in the context of weakly dispersive and fully nonlinear gravity waves is contained in a recent paper of Miles & Salmon (1985). For steady flow confined to the (x, z)-plane, the first of the two Boussinesq equations is the same as (2.2*a*) and the second of the Boussinesq equations can be reduced to a Bernoulli integral of the form

$$\tfrac{1}{2}u^2 + g(h-h_0) = \tfrac{1}{2}u_0^2. \tag{2.5}$$

Equation (2.5) may be compared with (2.3) after setting $H = 0$ and $\hat{p} = 0$ everywhere. It is clear that for steady flow the Boussinesq equations do not contain any part of the nonlinear terms represented by the function χ in (2.4), although for unsteady flow the second of the Boussinesq equations does possess a nonlinear term which vanishes in the case of steady flow. It can be easily verified that the Boussinesq equations do not admit non-trivial steady-state solutions; and hence, for steady flow, admit no wave-like solution.

Another system of equations attributed to Saint-Venant, which are often used in the discussion of open-channel-flow problems, are similar to those of Boussinesq. In fact, for steady flow, the first of the two Saint-Venant equations is the same as (2.2*a*) and the second is similar to (2.5) with the second term on the left-hand side of (2.5) replaced by $g(h-h_0+H)$ with $H = H(x)$ representing the effect of variable bottom. A generalization of Saint-Venant equations has been given by Dressler (1978) and an alternative derivation of Dressler's development is contained in a paper of Sivakumaran, Hosking & Tingsanchali (1981), who cite additional related references. As in the case of the Boussinesq equations, neither the Saint-Venant equations nor their generalizations (Sivakumaran *et al.* 1981) admit wave-like solutions in steady flow. It may be noted, however, that if higher-order terms are retained in the derivation of Saint-Venant equations, then a steady periodic solution is possible.

Before closing this section, it should be noted that the expressions for the pressure $\bar{p}$ at the bottom surface $z = H$ and the expression for the pressure p (which can be identified as the integrated pressure of the three-dimensional theory) in the theory of Green & Naghdi (1976*a*, *b*, 1977), with the use of the equations of motion can be expressed in terms of h and h_x. We do not record these expressions here, but note for later reference that they depend also on the pressure $\hat{p}$ at the top surface, the acceleration of gravity, as well as the effect of the bottom topography in terms of H and its derivatives.

3. Formulation of the problem: the upstream flow

A statement of the problem under consideration is given in the opening paragraph of §1. With reference to figure 1, we take the origin 0 of the (x, y)-coordinate axes to coincide with a convenient reference point within the transition region occupied by the obstacle (for example the point corresponding to the maximum height depicted in figure 1). Furthermore, we denote the points on the left and right of the origin, which demark the transition between the level bottoms and the region occupied by the obstacle, by $x = x_1$ and $x = x_2$. The physical region of space then conveniently separates into three parts: (i) the upstream region $-\infty < x \leqslant x_1$ (labelled I in figure 1),

(ii) the transition region $x_1 \leqslant x \leqslant x_2$ (II in figure 1) and (iii) the downstream region $x_2 \leqslant x < \infty$ (III in figure 1). It follows that (in the absence of surface tension) the pressure $\hat{p}$ at the top surface equals the atmospheric pressure p_0 in regions I and III, while the pressure $\bar{p}$ at the bottom surface is to be determined for all three regions.

If it is assumed that there are no discontinuities in mass and energy, i.e. if there is no gain or loss in mass or energy at any point of the physical region of space under consideration, (2.3) is valid in each of the three regions I, II, III, with the constant Q in (2.4) specified by (2.2*b*). The truth of this statement can easily be proved with the help of the appropriate jump conditions (see Appendix of Naghdi & Rubin 1981) for a directed fluid sheet.

We now turn our attention to a brief discussion of the upstream flow, the conditions for which may be summarized as:

$$\text{Region I } (-\infty < x \leqslant x_1)\colon \quad \hat{p} = p_0,\ \bar{p} \text{ to be determined}, \quad H = 0. \tag{3.1}$$

Since $H = H_x = H_{xx} = 0$ in the interval $-\infty < x \leqslant x_1$, for region I the differential equation (2.3) after substitution for u in terms of h from (2*a*) reduces to

$$hh_{xx} - \tfrac{1}{2}h_x^2 + f(h) = 0, \quad f(h) = \frac{3h^3}{F^2h_0^3} - \frac{3}{2}\left(1 + \frac{2}{F^2}\right)\left(\frac{h}{h_0}\right)^2 + \frac{3}{2}, \tag{3.2a, b}$$

where in obtaining (3.2) we have set both $\hat{p}$ and $\hat{p}_\infty$ equal to the constant atmospheric pressure p_0 and taken this to be equal to zero without loss in generality, and where F denotes the far-upstream Froude number defined by (1.1).

By a standard procedure (3.2*a*) can be integrated once more to yield

$$h_x^2 + \frac{3}{F^2}\left(\frac{h}{h_0}\right)^3 - 3\left(1 + \frac{2}{F^2}\right)\left(\frac{h}{h_0}\right)^2 + \frac{6S_1}{F^2gh_0^2}\left(\frac{h}{h_0}\right) - 3 = 0, \tag{3.3}$$

where S_1 is a constant of integration. Again the constant S_1 can be determined from the far-upstream conditions as $x \rightarrow -\infty$ and is given by

$$S_1 = \tfrac{1}{2}F^2gh_0^2\left(2 + \frac{1}{F^2}\right). \tag{3.4}$$

Clearly, with the use of (3.4), the differential equation (3.3) can be written as

$$h_x^2 = \frac{3}{F^2h_0^3}(h - h_0)^2(F^2h_0 - h). \tag{3.5}$$

The solutions of (3.5) in region I depend on the value of the far-upstream Froude number F and are well known for the full range of values of F^2 (< 1, $= 1$, > 1). In particular, we recall here that if $F^2 < 1$, the only possible steady-state solution of (3.5) subject to the far-upstream conditions stated in (2.1) is

$$h = h_0 = \text{const.} \tag{3.6}$$

[Results of the type which follow from (3.3) and (3.5) are well known and have been discussed previously by Benjamin & Lighthill 1954 and Benjamin 1956].

4. The downstream flow

We postpone consideration of the flow in region II and proceed now to some developments pertaining to the nature of the solution in region III downstream from the obstacle. As indicated earlier, the differential equation for the wave height h is again given by (3.2*a*). This can be integrated to yield an equation similar in form to (3.3), namely

$$h_x^2+q(h)=0, \quad q(h)=\frac{3}{F^2}\left(\frac{h}{h_0}\right)^3-3\left(1+\frac{2}{F^2}\right)\left(\frac{h}{h_0}\right)^2+\frac{6S_3}{F^2gh_0^2}\left(\frac{h}{h_0}\right)-3, \quad (4.1a, b)$$

where S_3 is another constant of integration. [The differential equation (4.1*a*) has the same form as Eq. (20) of Benjamin & Lighthill 1954]. Elimination of h_x^2 between (4.1*a*) and (3.2*a*), together with the use of (3.2*b*), results in the following interesting relation:

$$h_{xx}+\frac{1}{2}\frac{\mathrm{d}q}{\mathrm{d}h}=0. \quad (4.2)$$

With the help of (4.1*a*, *b*) and (4.2), it can be shown by a standard argument that when $q(h)$ has only one positive real root $h = h_2$ (say), which is not a double or triple root, then h must vanish at some point in the region $x_2 < x < \infty$ and a steady-state solution is not possible downstream. For later reference, we observe here that in the case of a level bottom ($H = H_x = H_{xx} = 0$) and when $\hat{p} = p_0 = 0$, it can be shown with the help of (4.1) and the basic equations of motion that

$$\bar{p}=\tfrac{1}{4}\rho^*\left[gh+\frac{6S_3}{h}-\frac{6Q^2}{h^2}\right], \quad p=\rho^*\left[S_3-\frac{Q^2}{h}\right], \quad (4.3a, b)$$

respectively.

Keeping in mind the observations made in the preceding paragraph, we now proceed to obtain some analytical results in region III for the downstream flow. We begin by examining the nature of the solution when the flow approaches a uniform flow far downstream, i.e. when h approaches a constant value. Since h_x and h_{xx} both approach zero far downstream, it follows from (3.2*a*) that far downstream h must approach a root of the function $f(h)$ given by (3.2*b*). It can be easily verified that the right-hand side of (3.2*b*) can be factorized in the form

$$f(h)=\frac{3}{F^2h_0^3}(h-h_0)(h-\gamma h_0)\left[h-\left(\frac{F^2}{2}-\gamma\right)h_0\right], \quad (4.4)$$

so that γh_0 is a root of (3.2*b*) and γ – a function of F^2 – is defined by

$$\gamma=\gamma(F^2)=\tfrac{1}{4}F^2\left[1+\left(1+\frac{8}{F^2}\right)^{\frac{1}{2}}\right]=\tfrac{1}{4}[F^2+(F^4+8F^2)^{\frac{1}{2}}]. \quad (4.5a, b)$$

We may now conclude that the quantity $\frac{1}{2}F^2-\gamma$ is necessarily negative, i.e.

$$\tfrac{1}{2}F^2-\gamma=\tfrac{1}{4}F^2\left[1-\left(1+\frac{8}{F^2}\right)^{\frac{1}{2}}\right]<0. \quad (4.6)$$

It then follows from $f(h) = 0$ with f given by (4.4) that, if h approaches a constant value, then either $h \to h_0$ as $x \to \infty$, or $h = \gamma h_0$ as $x \to \infty$. Moreover, it can be shown that for the full range of values of the Froude number ($F^2 < 1, > 1, = 1$), γ must satisfy the conditions

$$F^2<\gamma<1 \quad \text{for } F^2<1, \quad 1<\gamma<F^2 \quad \text{for } F^2>1, \quad \gamma=1 \quad \text{for } F^2=1.$$

In the remainder of this section, we discuss in some detail the character of the downstream flow only for subcritical values of the far-upstream Froude number, i.e. for $F^2 < 1$.

4.1. *Solution for downstream flow when* $F^2 < 1$

If the downstream flow approaches uniform flow, with equilibrium height h_0, then the constant S_3 can be evaluated from (4.1*b*) by setting $h_x = 0$ and $h = h_0$, and is given by

$$S_3 = \tfrac{1}{2}F^2 g h_0^2\left(2+\frac{1}{F^2}\right). \tag{4.7}$$

Since the above value of S_3 is identical to S_1 in (3.4), (4.1) in this case reduces to (3.5). Moreover, since $F^2 < 1$, we again obtain a solution given by (3.6), i.e. $h = h_0$ everywhere for $x > x_2$. On the other hand, if the downstream flow approaches a uniform flow with equilibrium height γh_0, then with $h_x = 0$ and $h = \gamma h_0$ the value of S_3 from (4.1) is found to be

$$S_3 = \frac{F^2 g h_0^2}{2\gamma}\left[1+\left(1+\frac{2}{F^2}\right)\gamma^2-\frac{\gamma^3}{F^2}\right]. \tag{4.8}$$

From the fact that $h = \gamma h_0$ is a root of $f(h) = 0$ with $f(h)$ given by (3.2*b*), it follows that γ must satisfy

$$\frac{\gamma^3}{F^2}-\frac{1}{2}\left(1+\frac{2}{F^2}\right)\gamma^2+\tfrac{1}{2} = 0, \tag{4.9}$$

with the help of which (4.8) can be rewritten as

$$S_3 = \frac{F^2 g h_0^2}{2\gamma}\left(2+\frac{\gamma^3}{F^2}\right). \tag{4.10}$$

Next, after substituting for S_3 from (4.10) into (4.1*b*) and making use of (4.9), the expression for $q(h)$ can be factorized in the form

$$q(h) = \frac{3}{F^2 h_0^3}(h-\gamma h_0)^2\left(h-\frac{F^2 h_0}{\gamma^2}\right). \tag{4.11}$$

With $q(h)$ written in the form (4.11), the differential equation (4.1*a*) assumes an identical form to (3.5) of region I if (h_0, F^2) in the latter are replaced by

$$\bar{h}_0 = \gamma h_0, \quad \bar{F}^2 = \frac{F^2}{\gamma^3}, \tag{4.12a, b}$$

respectively. We may refer to $\bar{h}_0$ as the far-downstream wave height and to $\bar{F}$ as the far-downstream Froude number. By direct calculation, it can be verified from (4.12*b*) and (4.5*a*) that

$$\bar{F}^2 = 64F^{-4}\left[1+\left(1+\frac{8}{F^2}\right)^{\frac{1}{2}}\right]^{-3} > 1 \quad \text{for } F^2 < 1. \tag{4.13}$$

Hence, the downstream flow is supercritical with $\bar{F} > 1$ for all subcritical far-upstream conditions. It follows that one solution of (4.1*a*) with $q(h)$ given by (4.11) is the trivial solution for which $h_x = 0$ and h takes the value $\bar{h}_0$ in (4.12*a*). Further, when $h_x \neq 0$ everywhere, two possible solutions of (4.1*a*) are

$$h = \bar{h}_0+\bar{h}_0\left(\frac{F^2}{\gamma^3}-1\right)\operatorname{sech}^2\left\{\left[\frac{3(F^2-\gamma^3)}{4F^2}\right]^{\frac{1}{2}}\left(\frac{x}{\bar{h}_0}+a_4\right)\right\} \tag{4.14a}$$

$$h = \bar{h}_0-\bar{h}_0\left(\frac{F^2}{\gamma^3}-1\right)\operatorname{csch}^2\left\{\left[\frac{3(F^2-\gamma^3)}{4F^2}\right]^{\frac{1}{2}}\left(\frac{x}{\bar{h}_0}+a_5\right)\right\} \tag{4.14b}$$

(both for $\bar{F}^2 > 1$),

where a_4 and a_5 are constants of integration to be determined from the continuity requirements at $x = x_2$. We note that (4.14*a*) is a solitary wave so that $h > \bar{h}_0$ everywhere and h_x can be either positive or negative, while bounded from above and below. On the other hand, (4.14*b*) represents a wave with monotonically increasing height so that $h < \bar{h}_0$ everywhere and h_x is always positive but unbounded. Thus, the form of the solution in region III for $S_3 = S_3^-$ can be established from the knowledge of h and h_x at the beginning of region III. These solutions have been utilized previously in different contexts, e.g. Benjamin (1956), Caulk (1976) and Naghdi & Rubin (1982).

Clearly the solution of (4.1*a*) depends on the characteristics of the function $q(h)$ and hence on the value of the constant S_3. Furthermore, we expect that the values of S_3 given by (4.7) and (4.10) represent respectively, the upper and lower bounds for S_3. Accordingly, a steady flow is possible in the downstream region only if S_3 is bounded from above and below by

$$S_3^- \leqslant S_3 \leqslant S_3^+, \tag{4.15}$$

where

$$S_3^+ = \text{Right-hand side of (4.7)}, \quad S_3^- = \text{Right-hand side of (4.10)}. \tag{4.16a, b}$$

This is indeed the case and it can be shown by either algebraic or geometrical argument that, for values of $S_3 > S_3^+$ and $S_3 < S_3^-$, the function $q(h)$ has only one positive real root which is not a double or triple root. As noted earlier, in this case, a steady flow is not possible since the waveheight h would vanish at some point $x_2 < x < \infty$. Finally, for intermediate values of $S_3^- < S_3 < S_3^+$, the function $q(h)$ has three positive roots (h_1, h_2, h_3), such that $h_1 > h_2 > h_3$,

$$\frac{F^2 h_0}{\gamma^2} > h_1 > h_0 > h_2 > \gamma h_0 > h_3 > F^2 h_0, \tag{4.17}$$

and then (4.1*a*) can be written as

$$h_x^2 + \frac{3}{F^2 h_0^3}(h-h_1)(h-h_2)(h-h_3) = 0. \tag{4.18}$$

Integration of (4.18) results in the following cnoidal wave solution (see p. 597 of Abramowitz & Stegun 1965):

$$h = h_2 + (h_1 - h_2)\,\mathrm{cn}^2\left\{\tfrac{1}{2}(h_1-h_3)^{\frac{1}{2}}\left(\frac{3}{F^2 h_0^3}\right)^{\frac{1}{2}}(x-a_6)\right\}, \tag{4.19}$$

where a_6 is the value of x at which $h = h_1$ and cn is the Jacobian elliptic function with modulus $m = (h_1-h_2)/(h_1-h_3)$.

It follows from the foregoing development that the solution for steady flow in region III is determined once the value of the constant S_3 is known. Thus, if the value of the wave height h and its derivative h_x are known at some point in region III such as at $x = x_2$, then the constant S_3 from (4.1*a*, *b*) is calculated to be

$$S_3 = \tfrac{1}{2}F^2 g h_0^3\left\{1 + \left(1+\frac{2}{F^2}\right)\left(\frac{h}{h_0}\right)^2 - \frac{h^3}{F^2 h_0^3} - \tfrac{1}{3}h_x^2\right\}\Big/ h. \tag{4.20}$$

This completes our discussion of the steady flow in region III for subcritical far-upstream Froude number $F^2 < 1$. As noted earlier, the corresponding developments for $F^2 > 1$ and $F^2 = 1$ can be presented similarly, but these will not be considered here.

5. Flow over a curved bed

As an application of the developments of §§2–4, subject to the far upstream condition that $F^2 < 1$, we consider a steady flow over a bottom irregularity, such as a bell-shaped hump. A problem of this kind is usually treated by an 'approximate' elementary theory in which the pressure at any point in the fluid is assumed to be hydrostatic (see, for example, §2.6 of Henderson 1966). In the context of the theory of a directed fluid sheet, this is equivalent to neglecting the effect of vertical inertia in the system of basic equations. With this approximation, the elementary theory cannot account for a cnoidal wave-like motion downstream from the obstacle. Here, we also call attention to a related problem concerning the upstream influence discussed by Benjamin (1970).

By way of additional background, we recall that recently Sivakumaran *et al.* (1983) have reported some experimental results for a steady shallow flow which becomes supercritical downstream from the obstacle and have used a set of equations, first derived by Dressler (1978) and later rederived in a different manner by them (Sivakumaran *et al.* 1981), for a comparison with their experiments. Here we make use of their experimental results in order to assess the predictive capability of the development of §4. [It may be noted that the theoretical development in Sivakumaran *et al.* (1981) is based on an approach which is different from the basic theory of §2 and our development in §4. It should be noted also that Sivakumaran *et al.* (1983) obtained very good agreement of the measured free-surface profile and the pressure at the bed with the use of Dressler's equations.]

Preliminary to our main objective in this section, it is of interest to demonstrate that, for a given curved bed of any shape with (finite) arbitrary height, not all possible combination of the far-upstream conditions (h_0, F^2) will lead to a downstream flow approaching a supercritical uniform flow. To this end, we first note that the pressure $\bar{p}$ acts along the normal to the curved bed, so that the force (due to the bottom pressure) over a small arclength $\mathrm{d}s$ of the bed profile and per unit width is $\bar{p}\,\mathrm{d}s$. It then follows that the horizontal component of this force is $\bar{p}H_x\,\mathrm{d}x$ and its resultant along the $\boldsymbol{e}_1$ direction, called drag and denoted by $\mathscr{D}$, is given by

$$-\mathscr{D} = -\int_{x_1}^{x_2} \bar{p}H_x\,\mathrm{d}x. \tag{5.1}$$

This represents the force (or drag) resisting the flow, or equivalently the resultant force arising from the presence of the curved bed acting on the fluid. Also, the minus sign preceding $\mathscr{D}$ in (5.1) is introduced for convenience, rendering $\mathscr{D}$ positive. Observing that the integrand in (5.1) occurs also in the x-component of the momentum equation (see also the remarks just before (2.1)), then with $\hat{p} = p_0 = 0$ and use of (4.3b) and its analogue in region I, we have the following expressions for the pressure p at points $x = x_1$, x_2:

$$p = \rho^*\left[S_1 - \frac{Q^2}{h}\right] \quad \text{(in region I)}, \tag{5.2a}$$

$$p = \rho^*\left[S_3 - \frac{Q^2}{H}\right] \quad \text{(in region III)}. \tag{5.2b}$$

Next, recalling again the x-component of the momentum equation and using (2.1), from (5.1) we obtain

$$-\mathscr{D} = \int_{x_1}^{x_2} [p + \rho^* Qu]_x\,\mathrm{d}x = \left[p + \rho^*\frac{Q^2}{h}\right]_{x_1}^{x_2},$$

which, with the help of (5.2 a, b) and after inserting the value S_1 given by (3.5), yields

$$S_3 = \tfrac{1}{2}F^2 g h_0^2\left(2+\frac{1}{F^2}\right)-\frac{\mathscr{D}}{\rho^*}. \tag{5.3}$$

Clearly, if the right-hand side of (5.3) yields a value S_3^+, then the downstream flow for $x > x_2$ will be a uniform flow with height h_0 everywhere. On the other hand, if the right-hand side of (5.3) yields a value S_3^-, then the downstream flow for $x > x_2$ will approach a supercritical uniform flow of height γh_0. Further, although $\bar{p}$ in (5.1) can be expressed in terms of h, H and their derivatives, in general it is not possible to carry out the integration analytically and obtain an expression for $\mathscr{D}$ in analytical form. However, it is evident that $\mathscr{D}$ must depend on the far-upstream conditions (h_0, F^2) and also on the geometry of the bottom topography. Thus, for a given curved bed only certain combinations of the far-upstream conditions (h_0, F^2) could lead to a downstream flow approaching a supercritical uniform flow, i.e. only particular combinations of (h_0, F^2) in steady flow would render the right-hand side of (5.3) equal to S_3^-.

In order to demonstrate the quantitative effect of a curved bed on the downstream flow, let the bottom surface of the fluid in the transition region II be specified by a bell-shaped hump (such as that depicted in figure 1) in the form

$$H = H_0 \exp(-bx^2) \quad (x_1 \leqslant x \leqslant x_2), \tag{5.4}$$

where H_0 and b are constants. For a complete solution of the problem for all $x(-\infty < x < \infty)$, the solution in regions I, II and III must be matched at the transition points x_1 and x_2. Such matching may be accomplished by using the standard jump conditions associated with the integral balance laws of the theory of a directed fluid sheet. However, for the purpose of this section, it will suffice to assume that the bottom surfaces of the fluid sheets are continuous and smooth, so that with the use of continuity arguments the matching can be effected by joining the solution in region II with those in regions I and III smoothly at the transition points. Strictly speaking, the right-hand side of (5.4) vanishes only as $x \to \pm\infty$. However, the exponential function $\exp(-bx^2)$ approaches zero very rapidly so that by choosing x_1 and x_2 (the boundary points of the transition region in figure 1) sufficiently far from the origin, the resulting error will be minimal in assuming that H, H_x, H_{xx} all vanish at x_1 and x_2.†

The experiments of Sivakumaran *et al.* (1983) were carried out over a curved bed specified by

$$H = H_0 \exp\left[-\frac{1}{2}\left(\frac{x}{24}\right)^2\right], \quad H_0 = 20 \text{ cm}, \quad b = \tfrac{1}{2}(24)^2. \tag{5.5a, b, c}$$

They give several values of the flow rate Q and total head E [or 'energy head' in the terminology of Sivakumaran *et al.* (1983)]‡ defined by

$$E = h_0 + \frac{Q^2}{2gh_0^2} = h_0(1+\tfrac{1}{2}F^2), \tag{5.6}$$

such that the flow downstream is a supercritical flow with diminishing height for all cases. For the purposes of comparison with experimental results of Sivakumaran *et al.*

† For example, if b is chosen to be $\frac{25}{72} \approx 0.347$, then at $x = \pm 10$ the exponential function $\exp(-bx^2)$ will have the value 8×10^{-16}.

‡ Sivakumaran *et al.*'s (1983) notation $h\cos\theta$, ζ, q, D and E correspond, respectively, to h, H, Q, h_0 and E of the present paper.

Experimental data of Sivakumaran *et al.* (1983)

F^2	h_0 (cm)	$\hat{h}_0 = h_0/H_0$
0.032	34.3	1.72
0.028	33.6	1.68
0.020	31.4	1.57
0.013	29.4	1.47
0.007	27.1	1.36

TABLE 1. Converted experimental data of Sivakumaran *et al.* (1983) for a steady flow over a bell-shaped hump characterized by (5.5) such that the downstream flow becomes supercritical with diminishing wave height approaching a uniform flow far upstream (this would be similar to a flow pattern depicted by AB_3 in figure 1).

(1983), it is convenient to convert their upstream data from Q and E to h_0 and F^2. In order to effect this conversion, we solve for h_0 from the first of (5.6) and then calculate the square of Froude number from the expression $F^2 = Q^2/gh_0^3$. The upstream measured values (after conversion) of F^2 and h_0 are listed in the first two columns of table 1. As will become evident presently, we calculate the normalized height h_0/H_0, where H_0 is the value given by (5.5*b*). The corresponding experimental values are recorded in the third column of table 1 and represent the minimum height (relative to the bottom surface) such that the flow becomes supercritical far downstream, i.e. $h \rightarrow \bar{h}_0$ as $x \rightarrow \infty$.

We recall that the differential equation for the free-surface wave height in region II is given by (2.3) after identifying $\hat{p}$ with the constant atmospheric pressure, which we set equal to zero. Then, after substitution for u in terms of h from (2.1), (2.3) becomes

$$h_{xx} - \frac{1}{2}\frac{h_x^2}{h} + \tfrac{3}{2}h\left(\frac{1}{h^2} - \frac{1}{h_0^2}\right) + \frac{3h}{F^2 h_0^3}(h + H - h_0) + \tfrac{3}{2}\left(H_{xx} + \frac{H_x^2}{h}\right) = 0, \tag{5.7}$$

where from (5.4) the derivatives H_x and H_{xx} in terms of H are given by

$$H_x = (-2bx)\,H, \quad H_{xx} = (4b^2x^2 - 2b)\,H. \tag{5.8a, b}$$

Since the far-upstream conditions are restricted to be subcritical ($F^2 < 1$), the only possible solution in region I ($x \leqslant x_1$) is $h = h_0$ everywhere. It then follows that in the present development the appropriate continuity conditions to be satisfied by the solution of (5.7) are

$$h = h_0, \quad h_x = 0 \quad \text{at } x = x_1. \tag{5.9}$$

Although the bottom profile specified by (5.4) is relatively simple and well behaved, it is still difficult to obtain an analytical solution and h must be determined numerically in region II. Since we are interested mainly in the effect of the size of the obstacle (e.g. its vertical height) on the downstream flow, it is desirable to give here some background on the expected effect of H_0 (the height of the bell-shaped hump) on the downstream flow. Recalling from §4 that the characteristics of the flow in region III are known once the constant S_3 is specified, we only need to determine the effect of H_0 on S_3 for steady flow calculated at $x = x_2$ from (4.20) for the admissible range of the far-upstream conditions (h_0, F^2). If $S_3 = S_3^+$ at x_2, then $h = h_0$ everywhere in the downstream region. For a relatively large value of h_0 (i.e. for $h_0 \gg H$), we expect the value of S_3 to be fairly close to S_3^+ and the flow will be approximately uniform. Actually, if the value of S_3 is very close, but not equal, to S_3^+ the two largest roots of $q(h)$ will be close to one another and the flow in region III will be represented by a cnoidal wave with a very small amplitude. As the value of h_0 decreases while F^2 remains fixed, we expect the difference between S_3 and S_3^+ to increase and also the amplitude of the cnoidal wave to become larger. Eventually, at some fixed value of $h_0 = h_0^*$, the value of the constant S_3 may coincide with S_3^-, where S_3^- is defined by (4.16*b*), and the flow downstream becomes supercritical with h approaching the value $\bar{h}_0 = \gamma h_0$ as $x \to \infty$. For values of $h_0 < h_0^*$, there is no solution for steady flow in the region III as the value of S_3, if it exists, becomes less than S_3^-. The foregoing discussion excludes, of course, a situation in which the flow of region II separates from the bottom surface (creating a cavity) or the wave height vanishes before the flow reaches the transition boundary at $x = x_2$. It follows that h_0^* may be taken as the minimum value of h_0, i.e.

$$h_0^* = \min \text{ of } h_0 \text{ when } S_3 = S_3^-. \tag{5.10}$$

Further, it is evident that the value of h_0^* in steady flow depends on the far-upstream Froude number and we expect that the higher the value of the far upstream u_0 (corresponding to the higher value of F) the higher will be the value of h_0^*.

In carrying out a numerical integration of (5.7), it is desirable to express all relevant quantities in non-dimensional form. For this purpose, we define

$$\hat{h} = \frac{h}{H_0}, \quad \hat{h}_0 = \frac{h_0}{H_0}, \quad \hat{H} = \frac{H}{H_0} \tag{5.11a, b, c}$$

$$\hat{H} = \exp(-\hat{b}\hat{x}^2), \quad \hat{x} = \frac{x}{H_0}, \quad \hat{b} = bH_0^2. \tag{5.12a, b, c}$$

With the use of (5.11) and (5.12), a differential equation of the same form as (5.7) results, together with the continuity requirements for the solution $\hat{h}$ similar to (5.9), but with variables such as h_x, H replaced with $\hat{h}_{\hat{x}}$, $\hat{H}$, etc. In this way, if we identify H with the profile (5.5*a*), then the non-dimensionalized $\hat{H}$ is

$$\hat{H} = \exp[-(\tfrac{25}{72})\hat{x}^2]. \tag{5.13}$$

Then, for a fixed value of $F^2 < 1$, we may calculate the quantities

$$\frac{2S_3^+}{F^2 g h_0^2} = \left(2 + \frac{1}{F^2}\right), \quad \frac{2S_3^-}{F^2 g h_0^2} = \frac{1}{\gamma}\frac{2+\gamma^3}{F^2}, \tag{5.14a, b}$$

which are, respectively, the upper and lower bound values for the constant $(2/F^2 g h_0^2) S_3$ [see (4.16*a*, *b*)] associated with a uniform flow and a supercritical flow approaching the height $h = \bar{h}_0$ far downstream. For a given value of $\hat{h}_0$ the non-dimensionalized differential equation for $\hat{h}$ is integrated numerically (with the aid of a computer program) from $\hat{x}_1$ to $\hat{x}_2$. Thus, in terms of $\hat{h}$ and $\hat{h}_{\hat{x}}$ known at $\hat{x}_2$ and recalling (4.20), the value of S_3 at $\hat{x}_2$ is given by

$$\left(\frac{2}{F^2 g h_0^2}\right)\hat{S}_3 = \left\{\left[1 + \left(1 + \frac{2}{F^2}\right)\left(\frac{\hat{h}}{\hat{h}_0}\right)^2 - \left(\frac{\hat{h}^3}{F^2 \hat{h}_0^3}\right) - \tfrac{1}{3}(\hat{h}_{\hat{x}})^2\right] \Big/ \left(\frac{\hat{h}}{\hat{h}_0}\right)\right\}\Bigg|_{\hat{x}=\hat{x}_2}, \tag{5.15a}$$

$$\hat{S}_3 = S_3|_{\hat{x}=\hat{x}_2}. \tag{5.15b}$$

Clearly, if the value of $\hat{S}_3$ is close to S_3^-, the flow downstream is close to a uniform flow with $h = h_0$. Whether or not this value of $\hat{S}_3$ at $\hat{x}_2$ leads to a supercritical flow can be ascertained by simply comparing $\hat{S}_3$ with the value S_3^- in (5.14*b*). If not, we proceed to determine a new value for $\hat{S}_3$ by repeating the integration process for the same fixed values of F^2 but decreasing $\hat{h}_0$ to a new value. If necessary, this process is repeated and $\hat{h}_0$ is decreased until the new value of $\hat{S}_3$ reaches the value S_3^- in (5.14*b*).

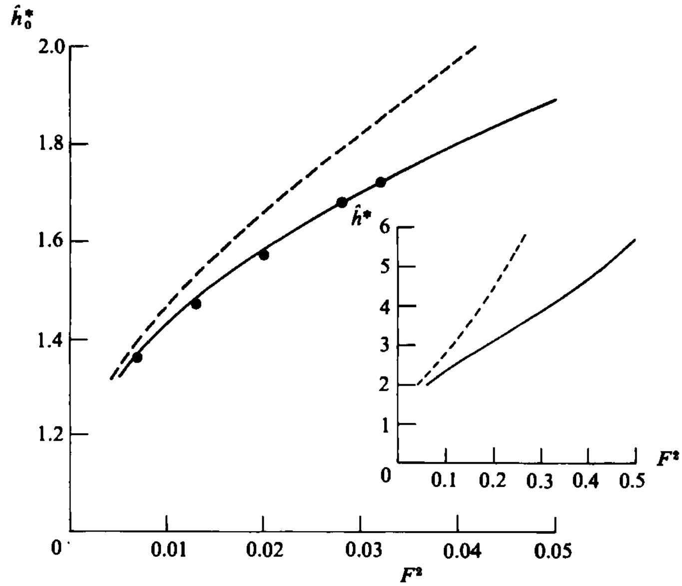

FIGURE 2. A plot of $\hat{h}_0^*$, representing the minimum upstream height h_0/H_0 (normalized with respect to the maximum height of the stationary bell-shaped hump on the bottom surface) below which no steady downstream flow is possible, *versus* the square of the Froude number F^2. The main part depicts a plot of $\hat{h}_0^*$ in the range $0.01 < F^2 < 0.05$ with data points —●— from experiments of Sivakumaran *et al.* (1983) recorded in table 1, while the inset shows the extension of variation of $\hat{h}_0^*$ in the range $0.1 < F^2 < 0.5$. The solid lines represent the predicted results calculated from (5.4), (5.5) and (5.15) and the dashed lines are calculated from the elementary theory referred to in the text.

This value of $\hat{h}_0$ (for given F^2) represents the minimum upstream height $\hat{h}_0^*$ that: (i) leads to a supercritical downstream flow; and (ii) below which no steady downstream flow is possible. In the actual numerical integration of the differential equation, it is very difficult to obtain a value for $\hat{h}_0^*$ such that $\hat{S}_3 = S_3^-$; but it is possible to adjust the process of decrementing $\hat{h}_0$ until $\hat{S}_3 \rightarrow S_3^-$. In the course of such calculations, it may happen that for a new value of $\hat{h}_0$ the height $\hat{h}$ vanishes at some point $\hat{x} < \hat{x}_2$ and $\hat{S}_3$ cannot be calculated. We take such a result to mean that $\hat{h}_0$ has been decreased to a value below $\hat{h}_0^*$. This difficulty in obtaining a precise value of $\hat{h}_0^*$ is well known and often occurs in the numerical process of finding a special solution which passes through a *critical* point, i.e. when the flow goes from subcritical to supercritical. In summary, while it is very difficult to obtain the value $\hat{h}_0^*$ such that $\hat{S}_3 = S_3^-$, by adjusting each increment in the process of calculations we may establish (i) a value of $\hat{h}_0$ just above $\hat{h}_0^*$ such that $\hat{S}_3 \rightarrow S_3^-$ as nearly as desired and (ii) a value of $\hat{h}_0$ just below $\hat{h}_0^*$ for which a steady flow does not exist. The actual value of $\hat{h}_0^*$ will be between these values, which can be made as close as desired. The process can be repeated for various values of F^2; a plot of the minimum height $\hat{h}_0^*$ *versus* F^2 is shown in figure 2 and represents the combinations of far-upstream conditions ($\hat{h}_0$, F^2) which render the downstream flow supercritical for fixed $\hat{H}$ in (5.13). Also shown in figure 2 are the experimental points of Sivakumaran *et al.* (1983), which are equivalent to the combination of columns 1 and 3 of table 1. For purposes of comparisons, the plot of minimum height *versus* F^2 calculated from the elementary theory is also given in figure 2. [Recall that the nature of the elementary theory was discussed in the first paragraph of this section.] It is interesting to observe the closeness of the curve of the elementary theory to that of the direct theory, especially in the range of very slow flow rate. However, such a close agreement is not likely in the presence of more drastic bottom irregularities (e.g. several humps of different heights in region II), as the elementary theory is not 'geometry dependent' and will give the same curve for any bottom surface with the same maximum height.

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REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. (ed.) 1965 *Handbook of Mathematical Functions with Formulas, Graphs and Mathematics Tables*. Dover.

BENJAMIN, T. B. 1956 On the flow in channels when rigid obstacles are placed in the stream. *J. Fluid Mech.* **1**, 227–248.

BENJAMIN, T. B. 1970 Upstream influence. *J. Fluid Mech.* **40**, 49–79.

BENJAMIN, T. B. & LIGHTHILL, M. J. 1954 On cnoidal waves and bores. *Proc. R. Soc. Lond.* A **224**, 448–460.

CAULK, D. A. 1976 On the problem of fluid flow under a sluice gate. *Intl J. Engng Sci.* **14**, 1115–1125.

DRESSLER, R. F. 1978 New nonlinear shallow-flow equations with curvature. *J. Hydraulic Res.* **16**, 205–220.

GREEN, A. E. & NAGHDI, P. M. 1976*a* Directed fluid sheets. *Proc. R. Soc. Lond.* A **347**, 447–473.

GREEN, A. E. & NAGHDI, P. M. 1976*b* A derivation of equations for wave propagation in water of variable depth. *J. Fluid Mech.* **78**, 237–246.

GREEN, A. E. & NAGHDI, P. M. 1977 Water waves in a nonhomogeneous incompressible fluid. *Trans. ASME* E: *J. Appl. Mech.* **44**, 523–528.

HENDERSON, F. M. 1966 *Open Channel Flow*. Macmillan.

MILES, J. & SALMON R. 1985 Weakly dispersive nonlinear gravity waves. *J. Fluid Mech.* **157**, 519–531.

NAGHDI, P. M. 1979 Fluid jets and fluid sheets: a direct formulation. In *Proc. 12th Symp. on Naval Hydrodynamics*, pp. 500–515. National Academy of Sciences, Washington, D.C.

NAGHDI, P. M. & RUBIN, M. B. 1981 On the transition to planing of a boat. *J. Fluid Mech.* **103**, 345–374.

NAGHDI, P. M. & RUBIN, M. B. 1982 The effect of curvature at the detachment point of a fluid sheet from a rigid boundary. *Phys. Fluids* **25**, 1110–1116.

SIVAKUMARAN, N. S., HOSKING, R. J. & TINGSANCHALI, T. 1981 Steady shallow flow over a spillway. *J. Fluid Mech.* **111**, 411–420.

SIVAKUMARAN, N. S., TINGSANCHALI, T. & HOSKING, R. J. 1983 Steady shallow flow over curved beds. *J. Fluid Mech.* **128**, 469–487.